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Minimum Expected Cost Control of a Remotely Piloted Vehicle

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A technique for designing constant gain feedback controllers for linear systems having uncertain or variable parameters is presented and demonstrated for a realistic design of a remotely piloted vehicle. This controller design technique—named Minimum Expected Cost Control—produces closed-loop system behavior which is acceptable for all values of the parameters within specified limits and is optimum in some overall sense. Only a small number of points in parameter space are explicitly considered in the design algorithm, making it computationally fast and inexpensive. The technique is used to design a constant-gain lateral autopilot for a remotely piloted vehicle that will fly at a wide range of altitudes and airspeeds. Both full and partial state feedback situations are considered.

Nomenclature

$A(\omega)$ = open-loop dynamics matrix $(n \times n)$
A_0 = nominal open-loop dynamics matrix $(n \times n)$
A_1 = sensitivity of open-loop dynamics matrix to $\omega_1 (n \times n)$
A_2 = sensitivity of open-loop dynamics matrix to $\omega_2 (n \times n)$
$B(\omega)$ = control distribution matrix $(n \times m)$
C = controller feedback gain matrix $(m \times m')$
$\mathcal{C}$ = set of controller feedback gain matrices which produce stable closed-loop system for all allowable parameter values
C_0 = initial estimate of controller feedback gain matrix $(m \times m')$
$E[\cdot]$ = expected value of
$H(\omega)$ = output distribution matrix $(m' \times n)$
$J[x(0),\omega,C]$ = performance index
$J^*[x(0),C]$ = expected value of J over all possible ω
$\hat{J}(C)$ = expected value of J^* over all possible $x(0)$
$L(\omega,C)$ = solution to algebraic Lyapunov equation $(n \times n)$
m = number of control variables
m' = number of measured output variables
N = number of points in parameter space used in algorithm
n = number of state variables
n' = number of uncertain (or slowly varying) parameters
p = vehicle roll rate
$p(\omega)$ = probability density function for ω
Q = state weighting matrix $(n \times n)$
R = control weighting matrix $(m \times m)$
$\mathcal{R}^n$ = n-dimensional real cartesian space
r = vehicle yaw rate
$S(\omega,C)$ = solution to algebraic Lyapunov equation $(n \times n)$
$S^*(C)$ = expected value of S over all possible $\omega (n \times n)$
t = independent variable
Tr = trace of
$u(t)$ = control vector $(m \times 1)$
V = true airspeed of vehicle
v = component of vehicle velocity parallel to pitch axis (sideslip velocity)
$x(t)$ = state vector $(n \times 1)$
X_0 = covariance of initial state $(n \times n)$
$y(t)$ = vector of measured outputs $(m' \times 1)$
δ_a = aileron deflection
δ_{a_c} = commanded aileron deflection
ρ = atmospheric density
ϕ = vehicle roll angle
Ω = region of parameter space in which parameters are known to lie
ω = vector of uncertain (or slowly varying) parameters referenced to their nominal values
$\omega^{(i)}$ = ith point in parameter space
ω_i = ith component of ω

Superscripts and Subscripts

T = transpose
nom = nominal

Introduction

USABLE methods for designing feedback controllers that will be insensitive to variations in the parameters of the physical system being controlled will become valuable tools for designers. There are always uncertainties—sometimes

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large ones—in the knowledge of physical constants of the dynamic system to be controlled. In addition, particular system parameters may vary naturally during normal system operation. In the design of aircraft control systems, for example, there are uncertainties about actuator dynamics, about aerodynamics, and about structural dynamics. Moreover, some of these will be quite different for different speeds and altitudes of flight.

The design of feedback controllers for systems with uncertain or variable parameters can be accomplished in several ways. Parameter uncertainty can often be reduced substantially through extensive testing or through use of real-time or non-real-time system identification techniques. Alternately, parameter uncertainties or variations may simply be accepted at their a priori levels, and a control system designed so as to be, in some sense, insensitive to parameter variations.

It is the latter approach which is investigated in this paper. In particular, a linear, constant gain feedback controller is sought for a linear system with an uncertain system matrix, control distribution matrix, and output distribution matrix, such that the closed-loop system behavior is 1) acceptable for all values of the uncertain parameters within specified limits, and 2) optimum in some overall sense. Considerable effort has been expended in this general area over the years, and a number of controller design methods have been devised (Refs. 1-18, for example). To date, no one technique has received widespread acceptance from control system designers.

In Ref. 1, a comparative assessment of seven such methods was made in the context of wing load alleviation for the C-5A, with uncertainties assumed to exist in dynamic pressure, structural damping and frequency, and an aerodynamic stability derivative. Most of the methods were found to be at least somewhat burdensome computationally, and most did not produce control system designs judged to be significant improvements over a standard linear-quadratic synthesis design[19] which assumes precisely known parameters. The uncertainty weighting[1] and minimax[2] techniques were judged to be generally superior to the other approaches.

In Ref. 9, continuous- and discrete-parameter versions of the minimum expected cost method,[7,8,10] the guaranteed cost control method,[11] the multistep guaranteed cost control method,[12] the minimax method,[2] the uncertainty weighting method,[1] and the standard linear-quadratic synthesis method[19] were compared. Two problems were considered, a second-order system with two uncertain parameters, and a fifth-order lateral autopilot for a rudderless remotely piloted vehicle (RPV) with an uncertain aerodynamic coefficient. Properties of the design techniques on which the comparison was based include closed-loop system performance at nominal and off-nominal parameter values, computational cost and complexity, ease of implementation in a real system, and generality of the parameter uncertainty which can be dealt with. A more detailed comparison of these methods was carried out in Ref. 14. A fifth-order lateral autopilot for a rudderless RPV with two uncertain aerodynamic coefficients was used in the comparison.

Based on the latter two studies, the discrete minimum expected cost and multistep guaranteed cost control methods appear to be the methods of choice (at least, among those tested) for solving the class of controller design problems considered. The minimax and uncertainty weighting methods, which were found in Ref. 1 to be superior to a number of other design methods, failed when applied to the fifth-order examples because there is no point in the specified region of parameter uncertainty with the desired minimax property. The guaranteed cost control method tended to produce relatively large controller feedback gains, relatively large control effort, and overdamped dominant closed-loop poles. The standard linear-quadratic synthesis method failed in both examples when the uncertain parameters deviated significantly from their nominal values (leading to an unstable system for part of the parameter range).

In this paper, one of the two methods of preference, the minimum discrete expected cost method, is examined in greater detail. A successful numerical implementation of this approach is described. The dependence of the control system performance upon certain aspects of the design procedure (such as the number of discrete points in parameter space used to approximate the region of parameter uncertainty) is investigated. The method is used to design a constant gain feedback controller to be used as the lateral autopilot for a rudderless RPV over a wide range of airspeed-altitude flight conditions.

Statement of the Control Problem

Using state space notation, one may describe the dynamics of a linear, time-invariant system with several uncertain, but constant, parameters by

$$\dot{x}(t)=A(\omega)x(t)+B(\omega)u(t) \qquad (t\geq 0) \tag{1}$$

$$y(t)=H(\omega)x(t) \tag{2}$$

The following assumptions are employed throughout this paper:

1) The vector ω is constant (or varies slowly) and lies somewhere within a closed bounded region $\Omega \subset \mathcal{R}^{n'}$.

2) $[A(\omega), B(\omega)]$ form a controllable (or stabilizable) pair for all $\omega\in\Omega$.

3) Control system performance is characterized by a quadratic cost functional J, defined as follows

$$J=\int_0^\infty (x^TQx+u^TRu)\,\mathrm{d}t \tag{3}$$

where Q is a constant positive semidefinite symmetric matrix and R is a constant positive definite symmetric matrix.

4) $[A(\omega), Q^{1/2}]$ form an observable (or detectable) pair for all $\omega\in\Omega$, where $Q^{1/2}$ denotes any square root of the matrix Q, i.e., $[Q^{1/2}]^TQ^{1/2}=Q$.

5) The distribution of ω within Ω is described by a probability density function $p(\omega)$, with the properties:

$$\int_\Omega p(\omega)\mathrm{d}\omega=1 \qquad p(\omega)\geq 0 \qquad (\omega\in\Omega) \tag{4}$$

The objective is to design a linear constant gain feedback controller of the form:

$$u(t)=-Cy(t) \qquad (t\in[0,\infty]) \tag{5}$$

which will produce acceptable closed-loop behavior for all $\omega\in\Omega$, $t\in[0,\infty]$. The matrix C is constant.

The Minimum Expected Cost Concept

The value of J, as given by Eq. (3), achieved using the constant-gain linear feedback control law (5), is functionally dependent upon the initial state $x(0)$, the uncertain parameter vector ω, and the gain matrix C. This dependence may be expressed as follows:

$$J[x(0),\ \omega,\ C]=x^T(0)S(\omega,\ C)x(0) \tag{6}$$

where $S(\omega, C)$ is the positive definite solution to the algebraic Lyapunov equation

$$\begin{aligned} &S(\omega,C)\,[A(\omega)-B(\omega)CH(\omega)] \\ &+[A(\omega)-B(\omega)CH(\omega)]^TS(\omega,C) \\ &+Q+H^T(\omega)C^TRCH(\omega)=0 \end{aligned} \tag{7}$$

This equation has a positive definite solution provided that $A(\omega)-B(\omega)CH(\omega)$ is a stability matrix, i.e., provided that its eigenvalues all have negative real parts.

Since a range of parameter values is of interest, it is convenient to adopt as a performance measure the expected value of J over $\omega\in\Omega$. Thus, define

$$J^*[x(0),C]=\int_{\omega\in\Omega} J[x(0),\omega,C]p(\omega)\mathrm{d}\omega=x^T(0)S^*(C)x(0) \tag{8}$$

where

$$S^*(C)=\int_{\omega\in\Omega} S(\omega,C)p(\omega)\mathrm{d}\omega \tag{9}$$

For J^* to be bounded, the feedback gain matrix C must produce a closed-loop system matrix which is stable for every $\omega\in\Omega$. General conditions regarding the existence of such a C are yet unknown. In many situations, such a gain matrix C can be found. In other situations, where the region Ω is simply too large, no such gain matrix exists. We shall assume here that there exists a non-empty subset $\mathcal{C}$ of $\mathcal{R}^{m\times m'}$ such that $A(\omega)-B(\omega)CH(\omega)$ is stable for all $\omega\in\Omega$ and all $C\in\Phi$.

If $J^*[x(0),C]$ is minimized with respect to C, the result is dependent upon $x(0)$. This dependence can be eliminated by first averaging $J^*[x(0),C]$ over all possible $x(0)$, i.e., by minimizing

$$\hat{J}(C)=\mathrm{Tr}[S^*(C)X_0] \tag{10}$$

where

$$X_0=\mathrm{E}[x(0)x^T(0)] \tag{11}$$

In Eq. (11) the expected value is taken with respect to the random vector $x(0)$. Clearly, $\hat{J}(C)$ depends on statistical knowledge of the initial conditions. If no statistical information is available, one might simply choose

$$X_0=I \tag{12}$$

which corresponds to assuming that $x(0)$ is equally likely to lie in any direction in $\mathcal{R}^n$.

The basic idea in the minimum expected cost controller design approach, as presented in Refs. 7 and 8, is to select the feedback gains C so as to minimize the quantity $\hat{J}(C)$. To do so, it is useful to have an expression for the gradient of $\hat{J}(C)$ with respect to C. Such an expression is given by[8,14]

$$\frac{\partial\hat{J}(C)}{\partial C}=2\{RC\mathrm{E}[H(\omega)L(\omega,C)H^T(\omega)] -\mathrm{E}[B^T(\omega)S(\omega,C)L(\omega,C)H^T(\omega)]\}\qquad (C\in\mathcal{C}) \tag{13}$$

where $L(\omega,C)$ is the positive definite solution to the Lyapunov equation

$$L(\omega,C)[A(\omega)-B(\omega)CH(\omega)]^T + [A(\omega)-B(\omega)CH(\omega)]L(\omega,C)=-X_0 \qquad (\omega\in\Omega,\ C\in\mathcal{C}) \tag{14}$$

The optimal linear-quadratic output feedback results of Levine and Athans[20] are readily obtained as special cases of Eqs. (7), (13), and (14), by requiring that ω be fixed. Conventional optimal linear-quadratic full state feedback results (as in Ref. 19, for example) can be obtained by further requiring that H be the identity matrix.

What has been presented to this point is a useful conceptualization of optimal linear-quadratic controller design techniques, as applied to systems with uncertain parameters. It is not, however, a practical controller design technique in the general form stated, due to the computational difficulty associated with evaluating and minimizing $\hat{J}(C)$. Each iteration of a numerical function minimization scheme requires one or more evaluations of $\hat{J}(C)$ and/or its gradient. Each such evaluation involves an integration over all $\omega\in\Omega$. The integrand, in turn, requires solution of one or two nth order matrix Lyapunov equations (Eqs. (7) and (14)) at each $\omega\in\Omega$, for its evaluation. These processes can be carried out analytically only in simple cases, such as a low order system expressed in phase variable canonical form, with a uniform probability density function.[7,8]

By assuming a discrete, rather than a continuous, probability function for ω, these numerical difficulties can be overcome. This approach is discussed in the next section.

Minimum Discrete Expected Cost Design Method

The computational obstacles mentioned above can be reduced to a manageable level, if the point of view is taken that the inclusion of all $\omega\in\Omega$ in the definition of $\hat{J}(C)$ is unnecessary in practical terms. If only a fairly small number of points in Ω, which are in some sense representative of the entire region, are considered (for example, the corners and selected intermediate points), much computational effort can be avoided at little cost in performance, as will be shown. In effect, the continuous probability density function $p(\omega)$ in Eq. (9) is replaced by a collection of Dirac delta functions throughout Ω, whose amplitudes add to unity.[9]

The following algorithm is suggested for the determination of feedback gains C which minimize $\hat{J}(C)$:

1) Select weighting matrices Q and R.

2) Select points $\omega^{(1)},\ldots,\omega^{(N)}\in\Omega$ to be used in approximating the integral in Eq. (9) by a finite sum. Select or evaluate corresponding probability mass functions.

3) Choose $C_0\in\mathcal{C}$ to initialize the minimization algorithm, by an alternate design technique, or other means. C_0 must be such that $[A(\omega^{(i)})-B(\omega^{(i)})\ C_0H(\omega^{(i)})]$ is a stability matrix for $i=1,\ldots,N$.

4) Use a numerical minimization technique (e.g., quasi-Newton) to find the value of C which minimizes $\hat{J}(C)$. Function and gradient evaluations require solution of Eqs. (7) and (14) for $\omega=\omega^{(i)}$, $i=1,\ldots,N$. These Lyapunov equations are solved by the algorithm of Bartels and Stewart.[21] Since the equations are adjoint, they may be solved simultaneously by means of a transformation suggested in Ref. 22. The gradient of $\hat{J}(C)$ with respect to C is then evaluated according to Eq. (13), where the expected value operation now reduces to a simple summation over $\omega^{(i)}$, $i=1,\ldots,N$.

5) Check the closed-loop system behavior over Ω using the feedback gains determined in step 4. If the results are satisfactory, stop. Otherwise, increase N and return to step 2.

The minimum discrete expected cost algorithm described above is similar in philosophy to the design concept described in Ref. 10. As was noted above, this algorithm has been applied with considerable success to fifth-order examples with one or two uncertain parameters.[9,14] Its performance was comparable to one other recently proposed controller design method and was superior to the others tested. Its usefulness is further demonstrated below in the context of another fifth-order, two parameter example.

The selection of the number of points (N) in parameter space to be used in characterizing the region Ω is discussed in the example below, and in somewhat greater depth in Ref. 14. In the work which has been carried out to date, once N has been selected, the points $\omega^{(1)},\ldots,\omega^{(N)}$ have generally been distributed uniformly in each dimension over the region Ω (a very simple procedure if the boundaries of Ω have no curvature) and weighted equally. If the discrete points are so chosen, the spacing between points decreases as N increases, reducing the likelihood of poor closed-loop system behavior in some portion of Ω in which none of the N points happen to lie. The computational cost, of course, increases as N increases, so that a tradeoff exists between accuracy of representation of Ω and computational cost.

The sensitivities of the feedback gains and the closed-loop system performance to N have generally been found to be modest.[9,14] This fact is demonstrated in the example considered below. A theoretical discussion as to why this is the case is presented in Ref. 14.

A drawback of the minimum discrete expected cost method relative to some other design methods for the same generic class of control problems is the fact that an initial gain matrix is required which produces a stable closed-loop system matrix for $\omega=\omega^{(i)}$, $i=1,...,N$. This initial gain matrix can be determined by means of an alternate design method such as the guaranteed cost control method. The latter method does not produce a very good final design, in many cases,[9,12,14] but does guarantee stability, and is therefore a useful starting point for the minimum discrete expected cost design. Two alternatives would be to make use of a mapping technique, such as that described in Ref. 23, to initialize the gain matrix, or to simply proceed by trial and error.

Design of a Fifth-Order Lateral Autopilot for a Remotely Piloted Vehicle

Consider a system having the form of Eqs. (1) and (2), with

$$x^T=[v,p,r,\phi,\delta_a] \qquad u=\delta_{a_c} \tag{15}$$

(See Nomenclature for definition of symbols.)

The matrices $A(\omega)$ and $B(\omega)$ are 5×5 and 5×1, respectively, where ω is a 2-vector. For simplicity, the aircraft dynamics have been modeled such that

$$A(\omega)=A_0+A_1\omega_1+A_2\omega_2 \tag{16}$$

where

$$A_0=\begin{bmatrix} -0.3031 & 40.58 & -672.8 & 31.87 & 0.0 \\ -0.1824 & -3.087 & 1.004 & 0.0 & 19.73 \\ 0.0079 & -0.201 & -0.4221 & 0.0 & 1.875 \\ 0.0 & 1.0 & 0.1365 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -20.0 \end{bmatrix} \tag{17}$$

$$A_1=\begin{bmatrix} -0.3031 & 40.58 & -672.8 & 0.0 & 0.0 \\ -0.1824 & -3.087 & 1.004 & 0.0 & 0.0 \\ 0.0079 & -0.201 & -0.4221 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{18}$$

$$A_2=\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 19.73 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.875 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{19}$$

The matrix $B(\omega)$ is given by

$$B(\omega)=\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 20.0 \end{bmatrix} \tag{20}$$

The aircraft nominally flies at an airspeed of 400 knots and at an altitude of 25,000 ft. However, it is desired that the autopilot perform satisfactorily at airspeeds ranging from 290 to 425 knots and at altitudes ranging from 20,000 to 36,000 ft. Thus, in this particular application, there are two parameters on which the system matrix A depends. These parameters are not uncertain, but rather are different for different flight conditions. The method outlined above for handling systems with parameters which are uncertain but bounded is equally applicable to the situation at hand, in which the parameters are easily estimated but vary within bounds during normal system operation. The only restriction is that these variations be slow relative to the time constants of the system.

A linear relationship between the system matrix and the variable parameters, as in Eq. (16), cannot be obtained if altitude and airspeed are treated directly as the variable parameters. However, if ρV and ρV^2 are treated as the variable parameters, then a linear relationship can be obtained. With the choice

$$\omega_1=\rho V/\rho_{\text{nom}}V_{\text{nom}}-1 \tag{21}$$

$$\omega_2=\rho V^2/\rho_{\text{nom}}V_{\text{nom}}^2-1 \tag{22}$$

the matrices A_1 and A_2 given by Eqs. (18) and (19) are obtained. Clearly, many elements of the matrix $A(\omega)$ are affected by the parameters ω_1 and ω_2. (A linear relationship between A and ω is not necessary for application of the minimum discrete expected cost design method—it is merely a convenience.)

The dynamics described above apply to a particular remotely piloted vehicle which has no rudder. A similar problem, involving a fixed altitude and airspeed, but an uncertain aerodynamic coefficient, was investigated in Refs. 9 and 12. An additional uncertain aerodynamic coefficient was included in Ref. 14.

The weighting matrices Q and R were chosen to be

$$Q=\begin{bmatrix} 0.14 & 0.94 & -15.58 & 0.52 & -0.41 \\ 0.94 & 16.48 & -273.87 & 12.60 & -0.57 \\ -15.58 & -273.87 & 4553.7 & -213.90 & 2.02 \\ 0.52 & 12.60 & -213.90 & 23.54 & 22.83 \\ -0.41 & -0.57 & 2.02 & 22.83 & 38.96 \end{bmatrix} \tag{23}$$

$$R=2500 \tag{24}$$

The nondiagonal state weighting matrix in Eq. (23) was constructed by means of a model matching procedure described in Ref. 12.

The region of parameter variation Ω is rectangular in airspeed-altitude space. (It is curvilinear in $\omega_1-\omega_2$ space.) The set of points $\omega^{(1)},...,\omega^{(N)}$ used to characterize Ω in the computational algorithm described above was selected by choosing the four corners of Ω and equally spaced (in each

Table 1 Controller feedback gains, minimized cost function, and CPU time as functions of N (full state feedback)

	Feedback gain on						Normalized	
N	v	p	r	ϕ	δ_a	$\hat{J}(C)$	$\hat{J}(C)$	CPU time, s
4	−.01544	.2922	1.4271	.1525	.1663	12,987	1.171	1.99
9	−.01350	.1795	1.5346	.05511	.1757	11,970	1.079	5.90
16	−.01266	.1557	1.5738	.03179	.1908	11,592	1.045	22.51
25	−.01163	.09010	1.4628	.03126	.1081	11,411	1.029	11.54
36	−.01155	.09836	1.5050	.01804	.09175	11,298	1.019	10.56
49	−.01127	.07832	1.5080	.02283	.08694	11,223	1.012	10.69
64	−.01121	.07608	1.5084	.02092	.08634	11,165	1.007	57.88
81	−.01113	.07252	1.5090	.02043	.08549	11,122	1.003	22.78
100	−.01106	.06954	1.5094	.02018	.08477	11,089	1.000	31.04
LQR[a]	−.0045	−.0678	1.333	.00952	.0638	∞	∞	0.79

[a] LQR = conventional linear-quadratic regulator.

Table 2 Closed-loop eigenvalues at various flight conditions for the standard linear quadratic regulator design and the minimum discrete expected cost design with $N=25$ (full state feedback)

Airspeed, knots	Altitude, ft	Closed-loop eigenvalues: Linear-quadratic regulator	Closed-loop eigenvalues: Minimum discrete expected cost
400	25,000	$-20.15,\ -3.38,\ -.523\pm3.74j,\ -.517$	$-15.75,\ -8.77,\ -.58\pm3.97j,\ -.287$
290	20,000	$-20.58,\ -2.91,\ -.313\pm3.22j,\ -.435$	$-18.98,\ -5.40,\ -.39\pm3.92j,\ -.280$
290	36,000	$-20.88,\ -1.52,\ +.012\pm1.86j,\ -.781$	$-20.53,\ -2.77,\ -.139\pm1.908j,\ -.468$
425	20,000	$-19.75,\ -4.40,\ -.75\pm4.77j,\ -.429$	$-12.58\pm4.646j,\ -.778\pm5.05j,\ -.248$
425	36,000	$-20.41,\ -2.19,\ -.29\pm2.65j,\ -.850$	$-18.07,\ -5.65,\ -.403\pm2.81j,\ -.393$

Table 3 Controller feedback gains for partial state feedback cases

	Feedback gain on					
Variables fed back	v	p	r	ϕ	δ_a	$\hat{J}(C)$
p	0.0	0.6076	0.0	0.0	0.0	20,576
p,r	0.0	1.193	2.650	0.0	0.0	13,320
p,ϕ	0.0	2.497	0.0	0.3559	0.0	14,618
v,p	0.01510	2.114	0.0	0.0	0.0	19,344
v,p,r	−0.01001	0.1175	1.407	0.0	0.0	11,401
v,p,ϕ	−0.02736	1.161	0.0	1.163	0.0	12,832
p,r,ϕ	0.0	1.178	2.663	−0.01648	0.0	13,319
v,p,r,ϕ	−0.01020	0.1252	1.406	0.01485	0.0	11,400

direction) intermediate points. In all cases considered, N was chosen to be a perfect square, and the airspeed and altitude ranges were each divided into $\sqrt{N}-1$ segments. The N points were weighted equally in each case.

Table 1 lists the feedback gains determined by the minimum discrete expected cost algorithm as functions of N, assuming $H(\omega)$ to be the 5×5 identity matrix. This corresponds to full state feedback. Also included in the table are the values of the minimized cost function $\hat{J}(C)$, as given by Eq. (10) (the matrix X_0 was chosen to be the $n\times n$ identity matrix), and the CPU time, in seconds, required to carry out the computations on Caltech's IBM 370/3032.

Table 1 demonstrates several important properties of the minimum discrete expected cost controller design method. It is readily seen that the cost function decreases monotonically as N increases but that the rate of decrease is not large. The cost function obtained using 16 points is only 5% larger than the cost function obtained using 100 points, for example. The changes in feedback gains, though larger in relative terms than the changes in the cost function, are also modest. Thus, one can conclude that (beyond some small number) the controller design is not terribly sensitive to the number of points in parameter space considered, so long as they are spaced uniformly and weighted equally, as was done here. This conclusion is consistent with the results of less extensive studies reported in Refs. 9 and 14. The monotonically decreasing behavior of the minimized cost function with increasing N is a general result, provided that $\mathrm{Tr}[S(\omega,C)]$ is convex in ω throughout Ω. Further arguments along these lines are presented in Ref. 14.

One would expect the CPU time used in executing the algorithm to increase monotonically with N. This is, in fact, the general trend, as is shown in Table 1, but the detailed behavior is somewhat erratic. The number of computations per iteration of the numerical minimization algorithm is almost directly proportional to N. However, the number of iterations required for the minimization algorithm to converge can vary substantially from one situation to another, producing deviations from the basic upward trend.

This same autopilot problem was solved using the conventional linear-quadratic regulator (LQR) synthesis approach, which assumes precisely known (and fixed) parameters. The resulting feedback gains are listed in the last line of Table 1. The closed-loop eigenvalues at the nominal and extreme flight conditions for this LQR design are listed in Table 2. The corresponding eigenvalues for the minimum discrete expected cost design with $N=25$ and full state feedback are also presented in Table 2. Note the instability occurring at one flight condition (low speed and high altitude) for the conventional linear quadratic regulator design, and note that the minimum discrete expected cost design is stable by an acceptable margin at all extreme points (Table 2).

One substantial advantage of the minimum discrete expected cost method over a number of the other methods mentioned in the Introduction is the fact that it can produce partial state feedback (i.e., output feedback, where rank $(H) < n$) designs just as easily as full state feedback designs. With many other design methods, such as the guaranteed cost control approaches,[9,11,12,14] this is not the case. A number of minimum discrete expected cost designs were carried out assuming various combinations of state variables available for feedback. The feedback gains obtained for these designs are presented in Table 3. Those state variables not available for feedback are indicated by the presence of zeroes in the gain matrix. In each case, N was taken to be 25, and the same gain set was used to initialize the numerical algorithm (this gain set was different from the initial gain set used in constructing Table 1). The required CPU time ranged from 8 seconds to 40 seconds, with the longer times associated with the cases in which three or four state variables were fed back.

The fifth state variable, δ_a, is probably the variable which one would least like to have to measure for feedback purposes. Comparison of Tables 1 and 3 indicates that its exclusion from the feedback law causes no increase in the minimum value of $\hat{J}(C)$. Thus, it can be eliminated from the feedback law without adversely affecting the control system performance. Elimination of feedback on ϕ, in addition, produces no significant change in the minimum value of $\hat{J}(C)$. Thus, feedback on v, p, and r is virtually indistinguishable from feedback on all five state variables. However, further elimination of state variables from the feedback law does produce a noticeable degradation in system performance.

Conclusion

The minimum discrete expected cost method for designing constant gain feedback controllers for linear systems with large parameter uncertainty has been examined in depth. A successful numerical implementation of this approach has been presented. It has been shown that a satisfactory controller design for a system with uncertain parameters can be achieved by explicitly considering only a small number of representative points in parameter space. This causes the computational cost of the method to be quite modest.

The practicality of the method has been demonstrated in the context of a lateral autopilot design for a remotely piloted vehicle—a fifth-order design with two variable parameters. (The method is applicable to situations in which some parameters are not unknown, but merely vary within prescribed bounds, so long as the variations are sufficiently slow.) The capability of the method to produce satisfactory partial state, as well as full state, feedback controllers has been demonstrated. It is possible to extend the minimum discrete expected cost method to the design of dynamic compensators (or observers), so as to allow estimation of those state variables for which direct measurements are not available.[24]

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